Determination of the total angular momentum of a paraxial beam

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The multipolar solution of the electromagnetic field is useful in light-matter interactions. Here is presented a method to control the superpositions of multipolar electromagnetic fields using paraxial beams. This is applied to orbital angular momentum paraxial modes, which allow us to calculate their total angular momentum. This quantity plays an important role in the control of electronic transitions in atoms and molecules. The results here can easily be extended to highly focused laser beams.

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I. INTRODUCTION

The transfer of angular momentum from photons to material particles is of paramount importance in light-matter interactions [1]. The angular momentum study of photon-atom interactions gives rise, for example, to the well known selection rules in atomic physics. In order to properly understand the angular momentum of a light field, one needs to use the spherical multipole solutions of Maxwell equations [2,3]. This set of solutions is highly nonparaxial and extend all over the space. This poses a problem when trying to explore multipolar interactions because normal experimental conditions use paraxial or near-paraxial beams.

Actually, for paraxial beams the work of Allen and coworkers [4] showed that the spin and the orbital angular momentum (SAM and OAM) can be controlled separately in a paraxial beam. This triggered a wide range of applications of the OAM of light covering areas as diverse as trapping and rotating microparticles [5], astrophysical measurements [6,7], or quantum information [8]. Also, recently there has been a renewed interest in the interaction of the OAM of light with atomic ensembles [9]. In the context of controlling atomic and molecular transitions with the OAM of paraxial beams, the subject has been studied by many authors [10–13]. Most authors restrict themselves to the lowest order of the multipolar interactions, i.e., the dipole approximation. In this paper, another point of view for this problem is presented. I extract the multipolar content of paraxial beams which will allow us to, for example, easily understand the atomic selection rules induced by paraxial beams. In this way, all the possible multipolar transitions are comprehensively studied and analyzed.

Another important application of the method presented here is the control of the amplitude and phase of the different multipolar fields with the transversal shape and the polarization of beams. This could be of importance when one wants to control specific atomic multipolar transitions, like in some quantum information implementations with ions [14].

Also, this approach allows us to answer one question of importance in the context of light-matter interactions: What is the total angular momentum of a paraxial beam? In a paraxial beam we can easily control one component of the angular momentum (i.e., the component parallel to the propagation direction) but can we also control the total angular momentum of the beam? The answer to this question is found by expressing the paraxial beam in the basis of eigenstates of the total angular momentum, i.e., the multipolar states.

Let me start by introducing the multipolar solution of an electromagnetic field. A general electromagnetic field in free space has two contributions to the angular momentum ($\mathbf{J}$): the spin part (hereby denoted by $\mathbf{S}$), which is related to the vectorial character of the field, and the OAM ($\mathbf{L}$), which is related to the spatial structure of the field [15]. Both parts are needed to generate rotations in space, and consequently only the combination of the two components plays a meaningful role in the rotational symmetries of an electromagnetic wave, i.e., $\mathbf{J}=\mathbf{L}+\mathbf{S}$. Multipolar modes are precisely a set of solutions to the Maxwell equations which are eigenvectors of the square of the total angular momentum J^2, one component of the total angular momentum, i.e., the z component, $J_z=L_z+S_z$ and the parity operator. The exact form of these fields can be found, for example, in [2], whose notation I will follow. In this paper, I will just use the properties of the vector potential of the multipolar monochromatic fields in the solenoidal gauge. The vector potentials of the set of multipolar fields is $\mathbf{A}_{jm}^{(x)}$, where $x=m$ represents the magnetic multipoles and $x=e$ the electric multipoles. Both classes of multipoles have eigenvalues $\mathbf{J}^2\mathbf{A}_{jm}^{(x)}=j(j+1)\mathbf{A}_{jm}^{(x)}$ and $J_z\mathbf{A}_{jm}^{(x)}=m\mathbf{A}_{jm}^{(x)}$, but magnetic and electric multipoles differ in their parity.

Up to now, the multipolar fields and the paraxial modes with OAM have been used separately in different regimes. In this paper I close this gap and calculate the multipolar content of any paraxial beam and in particular of the OAM modes. This approach has already been tried numerically [16] with results which are compatible with those provided here. I present a fully analytical multipolar expansion of paraxial beams, which helps us to understand the relationship between the OAM of paraxial beams and the total electromagnetic angular momentum. In order to do so, I will provide a fully cylindrically symmetric solution of the electromagnetic field [17]. I will show that this solution is identical to the usual OAM modes in the limit of the paraxial approximation. Finally, I will expand this cylindrically symmetric

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solution into multipolar modes, I will show some examples, and discuss some possible applications.

II. MULTIPOLAR DECOMPOSITION OF ELECTROMAGNETIC WAVES

In order to fulfill this program, I will first use an electromagnetic field which consists of a superposition of rotated circularly polarized plane waves:

$$\mathbf{A} = \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\varphi g(\theta,\varphi) R(\theta,\varphi) \mathbf{e}_p \exp(ikz). \quad (1)$$

Here, the original plane wave propagates along the z direction and is right (left) handed polarized when $p=1$ ($p=-1$). The altitude and azimuthal rotation angles are given, respectively, by (θ,φ). The operator R rotates the vector field in the usual manner: $R(\theta,\varphi)\mathbf{e}_p \exp(ikz) = M^{-1}\mathbf{e}_p \exp[ik\hat{\mathbf{z}}\cdot(M\mathbf{r})]$. This rotated plane wave is, of course, another plane wave propagating in the direction $\mathbf{k}' = \sin(\theta)\cos(\varphi)\mathbf{x} + \sin(\theta)\sin(\varphi)\mathbf{y} + \cos(\theta)\mathbf{z}$, with a circular polarization perpendicular to the propagation direction. Then, the sum (1) is an exact solution of the Maxwell equations. In particular, it can be used to describe beams highly focused with aplanatic lenses [18]. The function $g(\theta,\varphi)$ controls the relative amplitudes of the plane waves.

Our next step consists in finding a family of fields using suitable functions $g(\theta,\varphi)$, which in the limit of paraxiality can be identified with the usual OAM modes. Let me then develop the rotation operator to the lowest orders in the angle θ to obtain

$$R(\theta,\varphi)\mathbf{e}_p e^{ikz} \approx e^{ik\theta\rho\cos(\phi-\varphi)+ikz-i[(k\theta)^2/2k]z}\left(e^{-ip\varphi}\mathbf{e}_p - \frac{\theta}{\sqrt{2}}\hat{\mathbf{z}}\right), \quad (2)$$

where cylindrical coordinates (ρ,ϕ,z) are used. Compare this result with a circularly polarized paraxial field of the kind $\mathbf{A} = \mathfrak{F}_l(\rho,z)\exp(ik_z z)\exp(il\phi)/\sqrt{2\pi}\mathbf{e}_p$, which can be written in its Fourier components as

$$\mathbf{A} = (2\pi)^{-1}\int_0^{\infty} k_r dk_r \int_0^{2\pi} d\phi_k f_l(k_r)\exp(il\phi_k) \times \exp[ik_r r\cos(\phi-\phi_k)]\exp[ikz - ik_r^2/(2k)z]\mathbf{e}_p$$

. Any paraxial beam can be expressed as a convenient superposition of this kind of modes [19].

It is clear from inspection of Eqs. (1) and (2) that if we do the correspondence

$$k_r = k\sin(\theta), \quad \phi_k = \varphi,$$

$$g(k_r/k,\phi_k) = (2\pi)^{-1}k^2 f_l(k_r)\exp[i(l+p)\phi_k], \quad (3)$$

then the two expansions of the paraxial beam, i.e., in Fourier components and in rotated plane waves, are one and the same within the paraxial approximation. Then, the set of functions g fulfilling Eq. (3) defines the sought-after family of fields. Note that, in general, the functions $f_l(k_r)$ do not have to be paraxial. In this way I have defined a fully cylindrically symmetric set of electromagnetic fields, which in the limit of paraxiality are identical to the OAM modes [17].

This identification is very useful because now I can use the expansion of a rotated plane wave into multipole waves [2,3] and perform the integral over the rotation parameters. The expansion reads

$$R(\theta,\varphi)\mathbf{e}_p e^{ikz} = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} i^j(2j+1)^{1/2} D^j_{mp}(\varphi,\theta,0)[\mathbf{A}^{(m)}_{jm} + ip\mathbf{A}^{(e)}_{jm}], \quad (4)$$

where $D^j_{mp}(\varphi,\theta,0)$ are the matrices of rotation for irreducible tensors of order j. This matrix can be expressed as $D^j_{mp}(\varphi,\theta,0) = \exp(-im\varphi)d^j_{mp}(\theta)$, where the reduced rotation matrices $d^j_{mp}(\theta)$ can be found in Ref. [2].

With Eqs. (3) and (4), the φ integral of Eq. (1) is immediate. Then, the following result holds:

$$\mathbf{A} = \sum_{j=\|l+p\|}^{\infty} i^j(2j+1)^{1/2} C_{jlp}[\mathbf{A}^{(m)}_{j(l+p)} + i\mathbf{A}^{(e)}_{j(l+p)}],$$

$$C_{jlp} = k^2\int_0^{\pi}\sin(\theta)d\theta d^j_{(l+p)p}(\theta) f_l[k\sin(\theta)]. \quad (5)$$

This equation is the main result of this paper. Note that Eq. (5) is a valid solution of the Maxwell equations, as $f_l(k_r)$ is not restricted to paraxial beams. The normalization of the function f_l implies that $\Sigma_j(2j+1)\|C_{jlp}\|^2 = 2$. It can be observed that all the multipolar beams in the superposition share the same value of $m=l+p$. This has the obvious meaning of summing the OAM and SAM in the paraxial approximation, but in the more general case it only implies that we have a family of fields with a well defined J_z value, i.e., a fully cylindrically symmetric solution. This solution is different from that presented in Refs. [4,20,21] and it does not have to fulfill the angular momentum equations derived there. Also, it is consistent with the results found in [16].

III. CONTROL OF MULTIPOLES WITH PARAXIAL WAVES

On the other hand, this result has another interesting interpretation. As multipole fields only depend on the wavelength, the multipolar content of a field can be controlled with only paraxial fields. For the sake of simplifying the equations, I will consider from now on that $k=1$, meaning that all the lengths are scaled by $1/k$. In order to find some analytical expressions for Eq. (5), I will step back a little bit and assume the paraxial approximation again. In this case, the amplitude of the multipolar field of order $(j, m=l+p)$ can be approximated by

$$C_{jlp} = (-)^l[(j+p)!(j-p)!(j+l+p)!(j-l-p)!]^{1/2} \times \sum_s \frac{(-)^s}{(j-l-p-s)!(j+p-s)!(s+l)!s!}\frac{M_{l+2s}\{f_l\}}{2^{l+2s}}, \quad (6)$$

where the number $M_a\{f_l\} = \int_0^{\infty} k_r dk_r k_r^a f_l(k_r)$ is the momentum of order a of the function f_l.

Let me now discuss an important class of functions, the Laguerre-Gaussian (LG) modes. The LG modes are defined by two indices: the OAM index, which I conveniently call l and an index stating the number of transversal nodes of the function, which I call q. The LG modes also depend on a continuous parameter w_0 which represents the transversal width (or beam width) of the mode. For this set of functions, the momenta M_a can be calculated analytically and then we can find an expression for Eq. (6) for the set of LG modes. I will detail now some examples.

The simplest case is that of a paraxial Gaussian beam with circular polarization. In this case the following result is obtained:

$$\mathbf{A} = \sum_{j=1}^{\infty} i^j (2j+1)^{1/2} C_{j0p}[\mathbf{A}_{jp}^{(m)} + ip\mathbf{A}_{jp}^{(e)}],$$

$$C_{j0p} = (-)^{j+1}(j+p)!(j-p)!\sqrt{\frac{1}{(j+\|p\|)!}} \times 2w_0^{-2j-1+2\|p\|} L_{j-2\|p\|}^{2\|p\|}(w_0^2). \quad (7)$$

Note the following features of this field. First, it is composed of multipolar solutions with different total angular momentum, but with the same projection of the angular momentum in the direction z, which is $+1$ or -1 depending on the polarization of the field. Also, in both cases the amplitudes of the multipolar components are the same, as C_{j0p} does not depend on the sign of p. In Fig. 1 I plot the different amplitudes of the multipolar components for $w_0 = 15$, and the field resulting when sum (7) is numerically performed.

Another interesting case is that of a Laguerre-Gaussian beam with $l = \pm 1$, $q = 0$, and circularly polarized $p = -l$. I have shown that, in the general case, all the multipolar fields have a component of J_z that is equal to $m = l + p$. Then, in this example the projection of the angular momentum in the z direction of both considered fields is zero, i.e., $m = l + p = 0$. The multipolar decomposition gives the following result:

$$\mathbf{A} = \sum_{j=1}^{\infty} i^j (2j+1)^{1/2} C_{j\pm 1 \mp 1}[\mathbf{A}_{j0}^{(m)} + ip\mathbf{A}_{j0}^{(e)}],$$

$$C_{j(l=\pm 1)(p=\mp 1)} = l(-)^j \sqrt{(j+1)!(j-1)!} \times 2^{1/2+1} w_0^{-2j} L_{j-1}^{1}(w_0^2). \quad (8)$$

Note that the weights of the multipoles of the two fields have the same magnitude and opposite signs: $C_{j+1-1} = -C_{j-1+1}$. An example of such fields is found in Fig. 2.

This particular example is very interesting as the two considered fields are decomposed in the same subset of multipolar modes: $\mathbf{A}_{j0}^{(m)}$ and $\mathbf{A}_{j0}^{(e)}$. This will always be the case of fields with opposite polarizations and $\Delta l = 2$. However, in this particular example, both fields share the same weights in the decomposition. This means that if one produces a superposition of the kind $\mathbf{A} = \alpha LG_{10}\mathbf{e}_{-1} + \beta LG_{-10}\mathbf{e}_{+1}$, a purely transversal electric field can be generated when $\alpha = -\beta$ or a pure magnetic one with $\alpha = \beta$. In other cases with opposite polarization and $\Delta l = 2$, one can produce a multipole field with a well defined parity just in a set of discrete values of j, by playing with the amplitudes of the superpositions and the beam widths of the two fields.

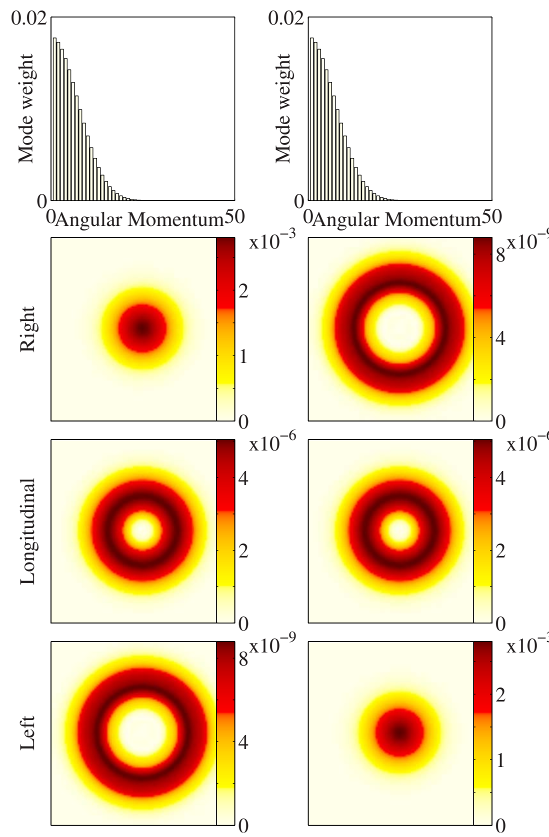

FIG. 1. (Color online) Multipolar decomposition of a Gaussian beam. First column: Right circular polarization ($p = +1$). Second column: Left circular polarization ($p = -1$). First row: weight of the multipolar components C_{j0p}^2. Second, third, and last rows represent the different components (respectively, right circular, longitudinal, and left circular) of the field at $z = 0$, after performing the summation of the multipole components. Note the different scales of the components.

IV. THE TOTAL ANGULAR MOMENTUM OF A PARAXIAL BEAM

An important property of the expansion (5) is that it easily allows us to calculate the total angular momentum of any beam of the kind (1) and, in particular, as mentioned earlier the total angular momentum of paraxial beams. In general, the radial density of total angular momentum per unit of energy of the field is the square root of the following quantity:

$$\langle \mathbf{j}^2 \rangle = \frac{\langle \mathbf{A}^* \mathbf{J}^2 \mathbf{A} \rangle}{\langle \mathbf{A}^* \mathbf{A} \rangle} = \frac{1}{2} \sum_j j(j+1)(2j+1)\|C_{jlp}\|^2. \quad (9)$$

The mean values are taken over a spherical shell of width dr. This expression can be applied to the LG beams described earlier. In Fig. 3(a) it is shown how the total angular momentum of several LG beams with different values of the azimuthal number l change with the width of the beam. In order

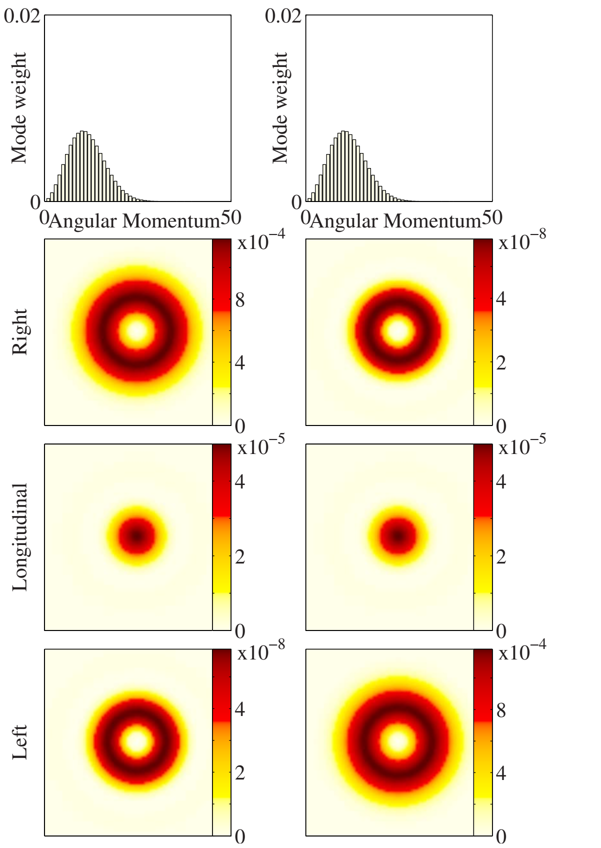

FIG. 2. (Color online) Multipolar decomposition of Laguerre-Gaussian beams. First column: Right circular polarization ($p=+1$) and azimuthal index negative ($l=-1$). Second column: Left circular polarization ($p=-1$) and azimuthal index positive ($l=+1$). First row: weight of the multipolar components: $C^2_{j\pm1\mp1}$. In this case $C_{j+1-1}=-C_{j-1+1}$. Second, third, and last rows are as in Fig. 1.

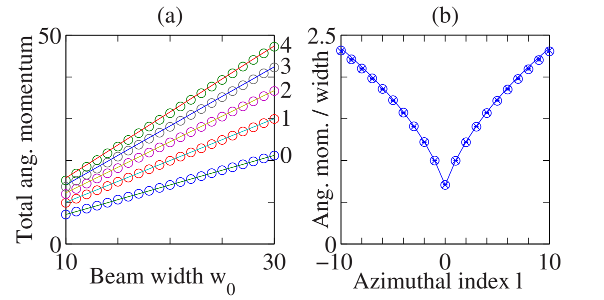

FIG. 3. (Color online) Total angular momentum of paraxial beams. (a) Dependence of the density of total angular momentum of a paraxial beam vs the beam width. Open circles represent the calculated value of the angular momentum for a LG beam with right circular polarization. The label of each set of data is the azimuthal index l of the LG beam. Continuous line follow the rule of $\sqrt{(\|l\|+1)/2}w_0$. (b) Steepness of the kind of curves shown in (a) with respect to the azimuthal index of the LG beam. The data are extracted by a linear regression of the calculated angular momentum. Open circles: Right circular polarization. Crosses: Left circular polarization. Error bars in the linear regression are smaller than the size of the spot. Continuous line: expected value of $\sqrt{(\|l\|+1)/2}$.

to avoid errors due to the paraxial approximation, I have used directly Eq. (5) to produce these results. It can be seen that the total angular momentum of the beam increases linearly with the width in the limit of large beam widths. Fits of the curves show that the total angular momentum of the beam follows the equation $\sqrt{\langle \mathbf{j}^2\rangle} \simeq \sqrt{(\|l\|+1)/2}w_0$. In Fig. 3(b) I present the result of the fits of the different curves with circles and the comparison with the value $\sqrt{(\|l\|+1)/2}$ inferred from this analysis. One can see that there is almost perfect agreement between values except for very low values of the width or very large values of l. The rule for the total angular momentum works otherwise perfectly when the beam is fully in the paraxial approximation. Note that the total angular momentum of a beam does not depend on its polarization or the sign of the azimuthal index. It is interesting to remark that from the results of Fig. 3 most of the angular momentum of a paraxial beam is distributed among the components perpendicular to the z axis.

V. APPLICATIONS

The examples above, and Eq. (5), demonstrate that we can control the superpositions of a set of multipolar fields by using paraxial beams. As considered above, multipolar fields are the solutions of the electromagnetic field one should use when treating spherically symmetric problems. For example, those problems where there is an exchange of angular momentum between the electromagnetic field and material particles are particularly well suited to be treated with multipolar expansions of the electromagnetic field. One such case is the electronic transitions in atoms or molecules. It is well known that multipolar transitions are increasingly more difficult to excite (or de-excite) for larger j's. This is why almost all the literature dedicated to the exchange of angular momentum (or more exactly OAM) treats the problem in the dipolar approximation. An easy calculation shows that the probability for a multipolar transition to occur scales as $(a/\lambda)^{2\Delta j}$, with a being the typical radius of the system involved, and Δj the multipolar transition [15].

Here I explore a different, but related, problem. Let us consider the case where it is needed to control a certain multipolar transition. This is the case, for example, when the system is in certain metastable states, as is the case of trapped Ca^+ ions in quantum information applications [14]. In those cases the rate of transition is low compared to dipole transitions, simply due to the small size of the ion, compared with the wavelength of the field. Nevertheless, one can still maximize the overlap of the laser beam with the electromagnetic multipole field associated with this transition. As in practical applications one has to deal with paraxial or near-paraxial beams, Eq. (6) can be used to engineer the shape and polarization of the control beam, to maximize the overlap with the desired transition. Importantly, this method allows us to control not only the total angular momentum exchange involved in a certain transition, but also the exchange of J_z. This can potentially lead to lower losses and errors in some applications, for example, in some quantum information protocols. Note that the approach followed here is fully equivalent to the usual approaches where the overlap of the electromagnetic field with the multipolar current of the material particle is calculated [2,15,22,23].

The results shown here could also be interesting in the field of nanophotonics where in some applications one needs to produce light which closely resembles the symmetry of a certain system, like, for example, with nanoantennas [24]. It is likely that our method allows one to introduce new degrees of freedom in this kind of applications, considering that our equation (5) is valid also for highly focused beams. In the same way, it could be used in optical tweezers applications to control the transfer of angular momentum to microparticles [5]. Also, the expansion shown here can be reversed and then one can create highly directional beams in applications where multipolar antennas are more easily available as, for example, in radio frequencies [25].

VI. CONCLUSION

In conclusion, this work closes an existing gap between the descriptions of angular momentum in paraxial and nonparaxial electromagnetic fields. I have demonstrated that a paraxial beam with OAM l and polarization p can be decomposed into multipolar modes with a fixed component of the angular momentum $m=l+p$ and different components of the total angular momentum j. I have provided analytical rules to calculate the weight of the superpositions for several interesting cases, thus allowing control over the superposition of the multipolar fields. These rules could be useful in fields where the interaction of light and matter has to be controlled with precision.

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